※正規分布の再生性

確率変数 X_1, X_2 がそれぞれ $N(\mu_1,\ \sigma_1^2)$, $N(\mu_2,\ \sigma_2^2)$ に従うとき，

$$X_1+X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

を示す．畳み込み (convolution) 積分を用いて

$$\begin{aligned}
& f_{X_1+X_2}(x) \\
= & \int_{-\infty}^{\infty} f_{X_1}(x-y) f_{X_2}(y) dy \quad \left(\int_{-\infty}^{\infty} f_{X_2}(x-y) f_{X_1}(y) dy \text{ としても同じなのは明らか.} \right) \\
= & \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{\{(x-y)-\mu_1\}^2}{2\sigma_1^2} \right] \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{\{y-\mu_2\}^2}{2\sigma_2^2} \right] dy \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2} \cdot (\sigma_2^2(x-y-\mu_1)^2 + \sigma_1^2(y-\mu_2)^2) \right] \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} \exp\left[-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}(y^2 + 2\frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} y) - \frac{1}{2\sigma_1^2\sigma_2^2}(\sigma_2^2x^2 - 2\mu_1\sigma_2^2 x + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2) \right] dy \\
= & \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2}(\sigma_2^2x^2 - 2\mu_1\sigma_2^2 x + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2) + \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2} \left(\frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} \right)^2 \right] dy \\
& \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} \exp\left[-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}(y + \frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2})^2 \right] dy \\
= & \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2}(\sigma_2^2x^2 - 2\mu_1\sigma_2^2 x + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2) + \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2} \left(\frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} \right)^2 \right] \\
& \cdot \frac{1}{\sqrt{(\sigma_1^2+\sigma_2^2)}} \int_{-\infty}^{\infty} \frac{\sqrt{(\sigma_1^2+\sigma_2^2)}}{\sqrt{2\pi}\sigma_1\sigma_2} \exp\left[-\frac{1}{2(\frac{\sigma_1\sigma_2}{\sqrt{(\sigma_1^2+\sigma_2^2)}})^2}(y + \frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2})^2 \right] dy \\
= & \frac{1}{\sqrt{2\pi}\sqrt{(\sigma_1^2+\sigma_2^2)}} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2}(\sigma_2^2x^2 - 2\mu_1\sigma_2^2 x + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2) + \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2} \left(\frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} \right)^2 \right] \\
= & \frac{1}{\sqrt{2\pi}\sqrt{(\sigma_1^2+\sigma_2^2)}} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2}(\sigma_2^2x^2 - 2\mu_1\sigma_2^2 x + \sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2) + \frac{\sigma_2^4x^2 - 2\sigma_2^2(\mu_1\sigma_2^2 - \mu_2\sigma_1^2)x + (\mu_1\sigma_2^2 - \mu_2\sigma_1^2)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)} \right] \\
= & \frac{1}{\sqrt{2\pi}\sqrt{(\sigma_1^2+\sigma_2^2)}} \exp\left[-\frac{(x-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)} \right]
\end{aligned}$$

従って

$$X_1+X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

が示せた．

なお

$$\int_{-\infty}^{\infty} \frac{\sqrt{(\sigma_1^2+\sigma_2^2)}}{\sqrt{2\pi}\sigma_1\sigma_2} \exp\left[-\frac{1}{2(\frac{\sigma_1\sigma_2}{\sqrt{(\sigma_1^2+\sigma_2^2)}})^2}(y + \frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2})^2 \right] dy = 1$$

が成り立つことを用いた．これは平均 $-\frac{-\sigma_2^2 x + \mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$，分散 $(\frac{\sigma_1\sigma_2}{\sqrt{(\sigma_1^2+\sigma_2^2)}})^2$ に従う正規分布の確率密度関数を全空間で積分したものであるから等号が成り立つ．